1.3 The Number Systems and the Real Number Line

Objectives

1. Classify Numbers
2. Plot Points on a Real Number Line
3. Use Inequalities to Order Real Numbers
4. Compute the Absolute Value of a Real Number

Are You Ready for This Section?

Before getting started, take the following readiness quiz. If you get a problem wrong, go back to the section cited and review the material.

R1. Write $\frac{5}{8}$ as a decimal. [Section 1.2, p. 14]

R2. Write $\frac{9}{11}$ as a decimal. [Section 1.2, p. 14]

Work Smart

The use of the word "real" to describe numbers leads us to question, "Are there 'nonreal' numbers?" The answer is yes. We use the word "imaginary" to describe nonreal numbers. Imaginary does not mean that these numbers are made up, however. We will discuss imaginary numbers later in the text.

This section discusses the *real number system.* We use the real numbers every day, so it is an idea that you are already familiar with. In short, real numbers are numbers that we use to count or measure things, such as 25 students in your class, 18.4 miles per gallon, or a $130 debt.

As we proceed through the text, we will deal with various types of numbers that are organized in *sets*. A **set** is a well-defined collection of objects. For example, we can identify the students enrolled in Elementary Algebra at your college as a set. The collection of numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 may also be identified as a set. If we let A represent this set of numbers, then we can write

$$A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

In this notation, braces { } are used to enclose the objects, or **elements,** in the set. A set with no elements in it is called an **empty set.** Empty sets are denoted by the symbol $\varnothing$ or { }.

EXAMPLE 1 Writing a Set

Write the set that represents the vowels.

Solution

The vowels are *a, e, i, o,* and *u*. If we let V represent this set, then

$$V = \{a, e, i, o, u\}$$

Quick ✓

1. Write the set that represents the first four positive, odd numbers.
2. Write the set that represents the states in the United States with names that begin with the letter A.
3. Write the set that represents the states in the United States with names that begin with the letter Z.

1 Classify Numbers

We will develop the real number system by looking at the history of numbers. The first types of numbers that humans worked with are the *natural numbers* or *counting numbers*. We introduced natural numbers in Section 1.2. We now present a formal definition using a set.

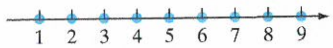
Definition

The **natural numbers**, or **counting numbers**, are the numbers in the set $\{1, 2, 3, \ldots\}$.

The three dots in this definition are called an *ellipsis* and indicate that the pattern continues indefinitely.

Ready?...Answers **R1.** 0.625 **R2.** 0.8181 . . . or $0.\overline{81}$

Figure 4
The natural numbers.



The natural numbers are often used to count things. For example, we can count the number of cars waiting at a Wendy's drive-thru. We can represent the counting numbers graphically using a number line. See Figure 4. The arrow on the right indicates the direction in which the numbers increase.

Since we do not count the number of cars waiting in the drive-thru by saying, "zero, one, two, three. . . ," zero is not a natural, or counting, number. When we add the number 0 to the set of counting numbers, we get the set of *whole numbers*.

Definition

The **whole numbers** are the numbers in the set $\{0, 1, 2, 3, \dots\}$.

Figure 5
The whole numbers.

Figure 5 represents the whole numbers on the number line. Notice that the set of natural numbers is included in the set of whole numbers.

By expanding the numbers to the left of zero on the number line, we obtain the set called *integers*.

Definition

The **integers** are the numbers in the set $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.

Figure 6
The integers.

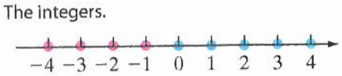


Figure 6 represents the integers on the number line. Notice that the whole numbers and natural numbers are included in the set of integers.

Integers are useful in many situations. For example, we could not discuss temperatures above 0°F (positive counting numbers) or below 0°F (negative counting numbers) without integers. A debt of 300 dollars can be represented as an integer by −300 dollars.

How can we represent a part of a whole, such as a part of last night's leftover pizza or part of a dollar? To address this problem, we enlarge our number system to include *rational numbers*.

In Words

A rational number is a number that can be expressed as a fraction where the numerator is any integer and the denominator is any nonzero integer.

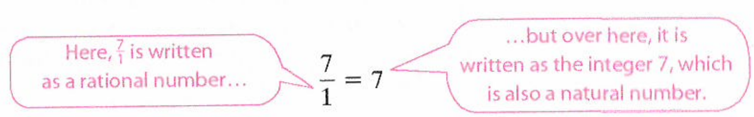
Definition

A **rational number** is a number that can be written in the form $\frac{p}{q}$, where p and q are integers. However, q cannot equal zero.

Examples of rational numbers are $\frac{2}{5}, \frac{5}{2}, \frac{0}{8}, -\frac{7}{9}$, and $\frac{31}{4}$. Because $\frac{p}{1} = p$ for any integer p, it follows that all integers are also rational numbers. For example, 7 is an integer, but it is also a rational number because it can be written as $\frac{7}{1}$. We illustrate this idea below.

Work Smart

Remember, all integers are also rational numbers. For example, $42 = \frac{42}{1}$.



In addition to representing rational numbers as fractions, we can also represent rational numbers in decimal form as either repeating decimals or terminating decimals. Table 1 shows various rational numbers in fraction form and decimal form.

Table 1

Fraction Form of Rational Number	Decimal Form of Rational Number	Terminating or Repeating Decimal
$\frac{7}{2}$	3.5	Terminating
$\frac{1}{3}$	$0.333\ldots = 0.\overline{3}$	Repeating
$-\frac{3}{8}$	-0.375	Terminating
$-\frac{15}{11}$	$-1.3636\ldots = -1.\overline{36}$	Repeating

The repeating decimal $0.\overline{3}$ and the terminating decimal -0.375 are rational numbers because they represent fractions (see Section 1.2).

Decimals that neither terminate nor repeat are called *irrational numbers.*

In Words
Numbers that cannot be written as the ratio of two integers are irrational.

Definition

An **irrational number** is a number that has a decimal representation that neither terminates nor repeats. Therefore, irrational numbers cannot be written as the quotient (ratio) of two integers.

An example of an irrational number is 1.343343334... because the decimal neither terminates nor repeats. Other examples of irrational numbers are the symbols $\sqrt{2}$, whose value is approximately 1.41421, and π, whose value is approximately 3.141593.

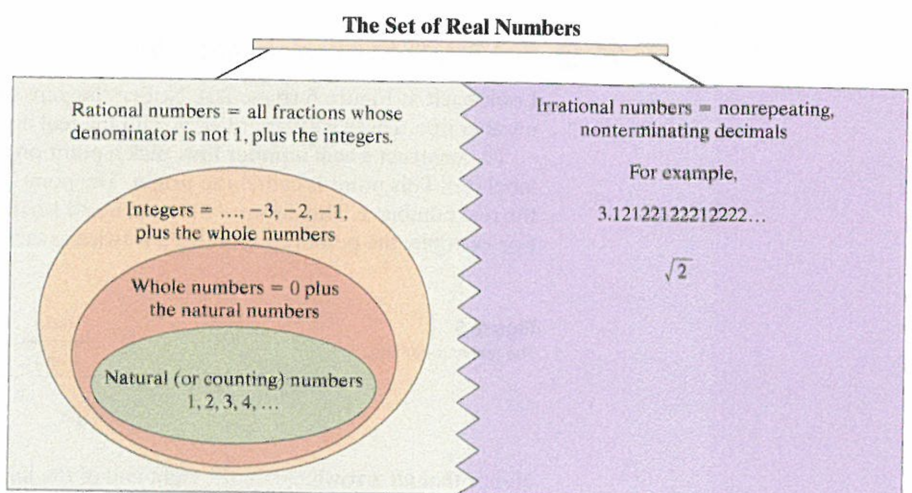
Now we are ready for a formal definition of the set of *real numbers.*

Definition

The set of rational numbers combined with the set of irrational numbers is called the set of **real numbers.**

Figure 7 shows the relationships among the various types of numbers. Note that the oval that represents the whole numbers surrounds the oval that represents the natural numbers. This means the set of whole numbers includes all the natural numbers.

Figure 7



The set of real numbers is composed of the set of rational numbers and the set of irrational numbers

EXAMPLE 2 Classifying Numbers in a Set

List the numbers in the set

$$\left\{9, -\frac{2}{7}, -4, 0, -4.010010001\ldots, 3.\overline{632}, 18.3737\ldots\right\}$$

that are

(a) Natural numbers
(b) Whole numbers
(c) Integers
(d) Rational numbers
(e) Irrational numbers
(f) Real numbers

Solution

(a) 9 is the only natural number.

(b) 0 and 9 are the whole numbers.

(c) 9, -4, and 0 are the integers.

(d) 9, $-\frac{2}{7}$, -4, 0, $3.\overline{632}$, and 18.3737 . . . are the rational numbers.

(e) $-4.010010001\ldots$ is the only irrational number because the decimal does not repeat, nor does it terminate.

(f) All the numbers listed are real numbers. Real numbers consist of rational numbers together with irrational numbers.

Quick ✓

4. *True or False* Every integer is a rational number.

5. Real numbers that can be represented with a terminating or repeating decimal are called ______ numbers.

In Problems 6–11, list the numbers in the set $\left\{\frac{11}{5}, -5, 12, 2.\overline{76}, 0, 2.737737773\ldots, \frac{18}{4}\right\}$ *that are*

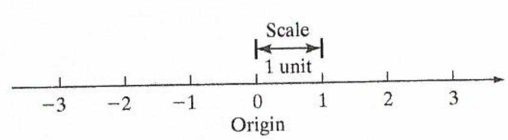
6. Natural numbers
7. Whole numbers
8. Integers
9. Rational numbers
10. Irrational numbers
11. Real numbers

2 Plot Points on a Real Number Line

Look back at Figure 6 (page 20). Notice the gaps between the integers plotted on the number line. These gaps are filled in with the real numbers that are not integers.

To construct a **real number line,** pick a point on a line somewhere in the center, and label it 0. This point is called the **origin.** The point 1 unit to the right of 0 corresponds to the real number 1. The distance between 0 and 1 determines the **scale** of the number line. For example, the point representing 2 is twice as far from 0 as 1 is. See Figure 8.

Figure 8
The real number line.



Notice that an arrowhead on the right end of the line indicates the direction in which the numbers increase. Points to the left of 0 correspond to the real numbers -1, -2, and so on.

Definition

The real number associated with a point P is called the **coordinate** of P.

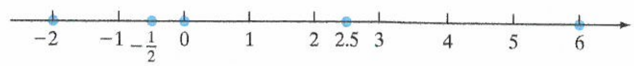
EXAMPLE 3 Plotting Points on a Real Number Line

On a real number line, label the points with coordinates $0, 6, -2, 2.5, -\frac{1}{2}$.

Solution

We draw a real number line and then plot the points. See Figure 9. Notice that 2.5 is midway between 2 and 3. Also notice that $-\frac{1}{2}$ is midway between -1 and 0.

Figure 9



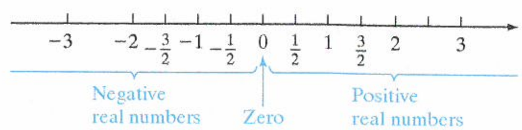
Quick ✓

12. The point on the real number line whose coordinate is 0 is called the _____.

13. On a real number line, label the points with coordinates $0, 3, -2, \frac{1}{2}$, and 3.5.

The real number line consists of three classes (or categories) of real numbers, as shown in Figure 10.

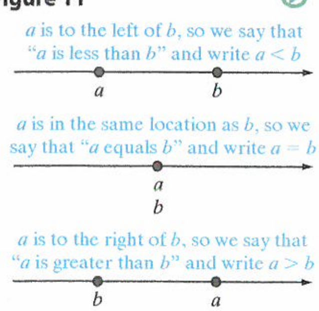
Figure 10



- The **negative real numbers** are the coordinates of points to the left of 0.
- The real number 0 is the coordinate of the origin.
- The **positive real numbers** are the coordinates of points to the right 0.

The **sign** of a number refers to whether the number is a positive or a negative real number. For example, the sign of -4 is negative and the sign of 100 is positive.

Figure 11



3 Use Inequalities to Order Real Numbers

Given two numbers (points) a and b, a must be to the left of b (denoted $a < b$) or the same as b (denoted $a = b$) or to the right of b (denoted $a > b$). See Figure 11.

If a is less than or equal to b, we write $a \leq b$. Similarly, $a \geq b$ means that a is greater than or equal to b. Collectively, the symbols $<, >, \leq$, and $\geq$ are called **inequality symbols.** The "arrowhead" in an inequality always points to the smaller number. For $3 < 5$, the "arrowhead" points to 3.

Note that $a < b$ and $b > a$ mean the same thing. For example, $2 < 3$ and $3 > 2$ mean the same thing. Do you see why?

EXAMPLE 4 Using Inequality Symbols

(a) We know that \$3 is less than \$7 and that 3 apples is fewer than 7 apples. Using the real number line, we say $3 < 7$ because the point whose coordinate is 3 lies to the left of the point whose coordinate is 7 on a real number line.

(b) Being \$2 in debt is not as bad as being \$5 in debt, so $-2 > -5$. Using the real number line, $-2 > -5$ because the point whose coordinate is -2 lies to the right of the point whose coordinate is -5 on a real number line.

(c) $2.7 > \frac{5}{2}$ because $\frac{5}{2} = 2.5$ and $2.7 > 2.5$.

(d) $\frac{5}{6} > \frac{4}{5}$ because $\frac{5}{6} = \frac{25}{30}$ and $\frac{4}{5} = \frac{24}{30}$, and 25 out of 30 parts is more than 24 out of 30 parts. We could also write $\frac{5}{6} = 0.8\overline{3}$ and $\frac{4}{5} = 0.8$. Since $0.8\overline{3}$ is greater than 0.80, $\frac{5}{6} > \frac{4}{5}$.

Work Smart

Write fractions with a common denominator or change fractions to decimals to compare the location of the numbers on the number line.

Quick ✓

14. The symbols $<, >, \leq, \geq$ are called ________ symbols.

In Problems 15–20, replace the question mark by $<, >,$ or $=$, *whichever is correct.*

15. 2 ? 9 **16.** -5 ? -3 **17.** $\frac{4}{5}$? $\frac{1}{2}$

18. $\frac{4}{7}$? 0.5 **19.** $\frac{4}{3}$? $\frac{20}{15}$ **20.** $-\frac{4}{3}$? $-\frac{5}{4}$

Based upon the discussion so far, we conclude that

$a > 0$ is equivalent to a is positive
$a < 0$ is equivalent to a is negative

We sometimes read $a > 0$ as "a is positive." If $a \geq 0$, then $a > 0$ or $a = 0$, so we may read this as "a is nonnegative" or "a is greater than or equal to zero."

4 Compute the Absolute Value of a Real Number

The real number line can be used to describe the concept of *absolute value*.

In Words

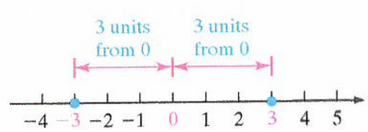
Think of absolute value as the number of units you must count to get from 0 to a number. The absolute value of a number can never be negative because it represents a distance.

Definition

The **absolute value** of a number a, written $|a|$, is the distance from 0 to a on a real number line.

For example, because the distance from 0 to 3 on a real number line is 3, the absolute value of 3, $|3|$, is 3. Because the distance from 0 to -3 on a real number line is 3, $|-3| = 3$. See Figure 12.

Figure 12



EXAMPLE 5 **Computing Absolute Value**

Evaluate each of the following:

(a) $|6|$ **(b)** $|-7|$ **(c)** $|0|$ **(d)** $-|-1.5|$

Solution

(a) $|6| = 6$ because the distance from 0 to 6 on a real number line is 6.
(b) $|-7| = 7$ because the distance from 0 to -7 on a real number line is 7.
(c) $|0| = 0$ because the distance from 0 to 0 on a real number line is 0.
(d) $-|-1.5| = -1.5$

Work Smart: Study Skills

Selected problem in the exercise sets are identified by a green color. For extra help, worked solutions to these problems are in MyMathLab.

Quick ✓

21. The distance from zero to a point on a real number line whose coordinate is a is called the _______ ____ of a.

In Problems 22–24, evaluate each expression.

22. $|-15|$ **23.** $\left|\frac{3}{4}\right|$ **24.** $-|-4|$

1.3 Exercises

MyMathLab® MathXL® PRACTICE

Exercise numbers in green have complete video solutions in MyMathLab

Problems **1–24** *are the* Quick✓s *that follow the* EXAMPLES.

Building Skills

In Problems 25–30, write each set. See Objective 1.

25. A is the set of *whole* numbers less than 5.

26. B is the set of *natural* numbers less than 25.

27. D is the set of *natural* numbers less than 5.

28. C is the set of *integers* between -6 and 4, not including -6 or 4.

29. E is the set of even *natural* numbers less than 1.

30. F is the set of *whole* numbers less than 0.

In Problems 31–36, list the elements in the set $\left\{-4, 3, -\frac{13}{2}, 0, 2.303003000\ldots\right\}$ *that are described. See Objective 1.*

31. natural numbers

32. whole numbers

33. integers

34. rational numbers

35. irrational numbers

36. real numbers

In Problems 37–42, list the elements in the set $\left\{-4.2, 3.\overline{5}, \pi, \frac{5}{5}\right\}$ *that are described. See Objective 1.*

37. real numbers

38. rational numbers

39. irrational numbers

40. integers

41. whole numbers

42. natural numbers

In Problems 43 and 44, plot the points in each set on a real number line. See Objective 2.

43. $\left\{0, \frac{3}{3}, -1.5, -2, \frac{4}{3}\right\}$

44. $\left\{\frac{3}{4}, \frac{0}{2}, -\frac{5}{4}, -0.5, 1.5\right\}$

In Problems 45–52, determine whether the statement is True or False. See Objective 3.

45. $-2 > -3$

46. $0 < -5$

47. $-6 \leq -6$

48. $-3 > -5$

49. $\frac{3}{2} = 1.5$

50. $4.7 = 4.\overline{7}$

51. $\pi = 3.14$

52. $\frac{1}{3} = 0.33$

In Problems 53–60, replace the ? with the correct symbol: >, <, =. See Objective 3.

53. $-1 \ ? \ 0$

54. $-8 \ ? \ -8.5$

55. $\frac{5}{8} \ ? \ \frac{6}{11}$

56. $\frac{5}{12} \ ? \ \frac{2}{3}$

57. $\frac{2}{9} \ ? \ 0.22$

58. $\frac{5}{11} \ ? \ 0.\overline{45}$

59. $\frac{42}{6} \ ? \ 7$

60. $\frac{3}{4} \ ? \ \frac{3}{5}$

In Problems 61–68, evaluate each expression. See Objective 4.

61. $|-12|$

62. $|-8|$

63. $|4|$

64. $|7|$

65. $\left|-\frac{3}{8}\right|$

66. $\left|-\frac{13}{9}\right|$

67. $-|-2.1|$

68. $-|-3.2|$

Mixed Practice

In Problems 69 and 70, (a) plot the points on a real number line, (b) write the numbers in ascending order, and (c) list the numbers that are (i) integers, (ii) rational numbers.

69. $\left\{\frac{3}{5}, -1, -\frac{1}{2}, 1, 3.5, |-7|, -4.5\right\}$

70. $\left\{8, -2, |-4|, -1.5, -\frac{4}{3}, 0, -\frac{15}{3}\right\}$

Applying the Concepts

In Problems 71–78, place a ✓ in the box if the given number belongs to that set.

		Natural	Whole	Integers	Rational	Irrational	Real
71.	-100						
72.	0						
73.	-10.5						
74.	$\sqrt{2}$						
75.	$\frac{75}{25}$						
76.	4						
77.	$7.56556555\ldots$						
78.	$6.\overline{45}$						

In Problems 79–88, determine whether the statement is True or False.

79. Every *whole* number is also an *integer*.

80. Every decimal number is a *rational* number.

81. There are numbers that are both *rational* and *irrational*.

82. 0 is a positive number.

83. Every *natural* number is also a *whole* number.

84. Every *integer* is also a *real* number.

85. Every terminating decimal is a *rational* number.

86. Some numbers in the form $\frac{p}{q}$, $q \neq 0$ are *integers*.

87. 0 is a nonnegative *integer*.

88. -1 is a nonpositive *integer*.

In Problems 89–94, name the set, or give the elements of the set, that matches each description.

89. nonterminating and nonrepeating decimals

90. nonnegative integers

91. the set of rational numbers combined with the set of irrational numbers

92. terminating or repeating decimals

93. numbers that are both nonnegative and nonpositive

94. numbers that are both negative and positive

Extending the Concepts

If every element of set A is also an element of set B, we say A is a **subset** *of B and we write $A \subseteq B$. In Problems 95–98, use this definition and the following sets to answer True or False to each statement.*

$$X = \{a, b, c, d, e\} \quad Y = \{c, e\} \quad Z = \{c, e, f\}$$

95. $Y \subseteq X$

96. $Z \subseteq Y$

97. $Y \subseteq Z$

98. $Z \subseteq X$

The **intersection** *of two sets is the set that contains the elements common to both A and B and is written $A \cap B$. The* **union** *of two sets is the set of all elements that are in either A or B and is written*

$A \cup B$. In Problems 99–104, write the elements of each set, using sets A, B, and C below.

$A = \{7, 8, 9, 10, 11, 12\}$ $B = \{10, 11, 12, 13, 14, 15\}$
$C = \{11, 12, 13, 14, 15\}$

99. $A \cup B$

100. $A \cap B$

101. $B \cup C$

102. $B \cap C$

103. $A \cap C$

104. $A \cup C$

105. If $A = \{\text{even integers}\}$ and $B = \{\text{whole numbers less than } 11\}$, find $A \cap B$.

106. If $X = \{48, 49, 50, \dots\}$ and $Y = \{60, 62, 64, \dots, 80\}$, find $X \cap Y$.

107. When writing subsets, it is important to be orderly when creating the list. Think of a pattern and then answer the following:

(a) List all possible subsets of set Z where $Z = \{1, 2, 3, 4\}$. *Hint:* The empty set is a subset of every set.

(b) How many subsets did you find?

108. Use the set $M = \{a, b, c\}$ to answer the following:

(a) List all possible subsets of M. *Hint:* The empty set is a subset of every set.

(b) How many subsets did you find?

(c) Determine a rule for finding the number of subsets of a set that has n elements.

Explaining the Concepts

109. Write a definition of "rational number" in your own words. Describe the characteristics to look for when deciding whether a number is in this set.

110. Write a definition of "irrational number" in your own words. Describe the characteristics to look for when deciding whether a number is in this set.

1.4 Adding, Subtracting, Multiplying, and Dividing Integers

Objectives

1. Add Integers
2. Determine the Additive Inverse of a Number
3. Subtract Integers
4. Multiply Integers
5. Divide Integers

Are You Ready for This Section?

Before getting started, take this readiness quiz. If you get a problem wrong, go back to the section cited and review the material.

R1. Write $\frac{16}{36}$ as a fraction in lowest terms. [Section 1.2, pp. 12–13]

R2. $|-5| = \_\_$. [Section 1.3, pp. 24–25]

In this section, we perform addition, subtraction, multiplication, and division, called **operations,** on integers. The symbols used in algebra for these operations are $+$, $-$, $\cdot$, and $/$, respectively. The results of these four operations are called the **sum, difference, product**, and **quotient,** respectively. Table 2 summarizes these ideas.

Table 2

Operation	Symbols	Words
Addition	$a + b$	Sum: *a* plus *b*
Subtraction	$a - b$	Difference: *a* minus *b*
Multiplication	$a \cdot b, (a) \cdot b, a \cdot (b), (a) \cdot (b), ab, (a)b, a(b), (a)(b)$	Product: *a* times *b*
Division	a/b or $\frac{a}{b}$	Quotient: *a* divided by *b*

Ready?...Answers **R1.** $\frac{4}{9}$ **R2.** 5

In algebra, we avoid using the multiplication sign × used in arithmetic. Instead, we multiply two expressions that are placed next to each other without an operation symbol, as in ab or that are in parentheses, as in $(a)(b)$, or we use · as in $a \cdot b$.

A **mixed number** is a whole number followed by a fraction. We do not use mixed numbers in algebra. When you see a mixed number, rewrite it as a fraction. Recall that to write $3\frac{2}{5}$ as a fraction, we multiply the whole number 3 by the denominator 5, obtaining 15, and then add this result to the numerator 2 to get 17. This result is the numerator of the fraction. The denominator remains 5. Thus

$$3\frac{2}{5} = \frac{17}{5} \leftarrow 3 \cdot 5 + 2$$

In algebra, mixed numbers are confusing because the lack of an operation symbol between two terms means multiplication. To avoid confusion, write $3\frac{2}{5}$ as 3.4 or as $\frac{17}{5}$.

Work Smart

Do not use mixed numbers in algebra.

1 Add Integers

Adding Integers with the Same Sign Using a Number Line

We will use a real number line to discover a pattern for adding integers. When we add a positive integer, we move to the right on the number line, and when we add a negative integer, we move to the left on the number line.

Remember, the *sign* of a number indicates whether the number is positive or negative. For example, the sign of 4 is positive, while the sign of -12 is negative. We will first consider adding integers with the same sign.

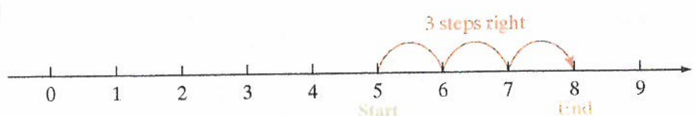
EXAMPLE 1 Adding Two Positive Integers Using a Number Line

Find the sum: $5 + 3$

Solution

We begin at 5 on the number line and move 3 spaces to the right, so $5 + 3 = 8$. See Figure 13.

Figure 13



EXAMPLE 2 Adding Two Negative Integers Using a Number Line

Find the sum: $-7 + (-4)$

Solution

We begin at -7 on the number line and move 4 spaces to the left, so $-7 + (-4) = -11$. See Figure 14.

Figure 14